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GROMOV'S COMPACTNESS OF PSEUDO-HOLOMORPHIC CURVES AND SYMPLECTIC GEOMETRY

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In a beautiful recent paper [6] M. Gromov has introduced into symplectic geometry some new and ingenious techniques. In particular he associates to a symplectic structure a distinguished family of almost complex structures. Certain pseudo-holomorphic curves, that is, certain maps of a Riemann surface into the symplectic manifold which are "holomorphic" with respect to one of these almost complex structures, then provide global invariants of the symplectic structure. In this way, for example, Gromov proves that the open round ball $B_R \subset \mathbb{R}^{2n}$ of radius R cannot be symplectically diffeomorphic to an open subset of an ε-neighborhood of a nondegenerate $(2n-2)$ plane in $\mathbb{R}^{2n}$, unless $R \leq \varepsilon$ (see [6, Theorem 0.3.A]). The difficult analytic step in this work is the proof of the existence of suitable pseudo-holomorphic curves. As in any existence proof for a nonlinear elliptic partial differential system (or equation) the proof divides into two parts: (i) a proof of the openness of the space of solutions (often this part is accomplished using an "inverse function theorem") and (ii) a proof of the closeness or compactness of the space of solutions. Gromov accomplishes (i) using an index theorem computation and the Sard-Smale Implicit Function Theorem. The proof of (ii) involves some of the most delicate and beautiful parts of the paper. Although the space of pseudo-holomorphic curves is not, in any suitable topology, compact, by enlarging the space to allow for certain singularities Gromov proves the required compactness theorem. The proof of this result in [6] is rather brief and quite difficult. Fortunately, Pansu [9] has written notes clarifying and expanding the details of this proof. The proof's most interesting feature is that it is entirely geometric and at no time refers to results in differential equations. This has the unfortunate consequence of making the paper difficult for many mathematicians. Both to make Gromov's compactness result more accessible and to unify it with the many other compactness results in differential geometry, we give a new proof of this result using some ideas in partial differential equations due to Schoen and Uhlenbeck [11]. This work appeared

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in preprint form under the title *A P.D.E. proof of Gromov's compactness of pseudo-holomorphic curves* and comprises Part I of this paper. Following the suggestion of the editors of this journal, in Part II we apply the compactness result of Part I and outline Gromov's proof of the symplectic rigidity of the open round ball.

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PART I
A P.D.E. PROOF OF GROMOV'S COMPACTNESS OF PSEUDO-HOLOMORPHIC CURVES

1. Formulation of the problem

In the introduction we briefly indicated the role Gromov's compactness of pseudo-holomorphic curves theorem plays in symplectic geometry. It is interesting to note that this result can be formulated and proved without reference to symplectic geometry. In this section following Pansu [9] and Gromov [6] we carry out this formulation.

Let (V, μ) be a compact Riemannian manifold equipped with an almost complex structure J, and M be a compact Riemann surface of genus g with complex structure J^M.

Definition. A map $f\colon (M, J^M) \to (V, J)$ is called a pseudo-holomorphic curve (or a J-holomorphic curve) if $df \circ J^M = J \circ df$.

Let $\{J_j\}$ be a sequence of almost complex structures on V, which is uniformly bounded in $C^{k+\alpha}$. Given $A > 0$, let $\{f_j\}$ be a sequence of C^k pseudo-holomorphic curves $f_j\colon (M, J^M) \to (V, J_j)$ of μ-area less than or equal to A.

Question 1. Does there exist a subsequence of the $\{f_j\}$ which converges in C^k to a pseudo-holomorphic curve $f_0\colon (M, J^M) \to (V, J_0)$?

Now suppose that the complex structures on M are varying. That is, suppose that J_j^M is a sequence of complex (or conformal) structures on M and that $\{f_j\}$ is a sequence of C^k pseudo-holomorphic curves $f_j\colon (M, J_j^M) \to (V, J_j)$ of μ-area less than or equal to A. The J_j are bounded as above.

Question 2. Does there exist a subsequence of the $\{f_j\}$ which converges in C^k to a pseudo-holomorphic curve $f_0\colon (M, J_0^M) \to (M, J_0)$?

It turns out that the answer to both questions is "no". However, the failure of compactness can be measured precisely and by enlarging the space suitable compactness theorems can be formulated. We begin with Question 1.

Theorem 1.1. *Let (V, μ) be a compact Riemannian manifold and let J_j be a sequence of almost complex structure uniformly bounded in $C^{k+\alpha}$, $k \geq 2$. Suppose that $f_j\colon (M, J^M) \to (V, J_j)$ is a sequence of C^k pseudo-holomorphic curves whose area is uniformly bounded by $A > 0$. Then there is a subsequence of the $\{f_j\}$ (still denoted by $\{f_j\}$) and a finite number of points $\{x_1 \cdots x_l\}$ such that $f_j \to f_\infty$ in $M - \bigcup_{i=1}^l U_{x_i}$ where U_{x_i} is any neighborhood of x_i. The convergence is in the C^k topology and f_∞ is a pseudo-holomorphic map of $M - \{x_1 \cdots x_l\}$ into V, for some almost complex structure J_∞.*

The negative answer to Question 1 results from the failure of C^k (or even C^1) convergence at the points $x_1 \cdots x_l$. This failure can be precisely understood. Let (ν be a hermitian metric on M and let $\|f_j\|_{C^1}$ denote the C^1 norm of f_j determined by ν and μ. Set $b_j = \sup_{D_\delta(x_1)} \|f_j\|_{C^1}$, where $\delta > 0$ is small. Let x_j be the point of $\overline{D_\delta(x_1)}$ where the value b_j is taken. Then as $j \to \infty$, $x_j \to x_1$ and $b_j \to \infty$. By rescaling the maps f_j in a smaller neighborhood of x_1 and using conformal invariance it can be shown that the blow-up of the f_j's results, in the limit, in the point x_1 being replaced by a pseudo-holomorphic curve $f_{\infty,1}\colon S^2 \to V$. A pseudo-holomorphic two-sphere "bubbles" up from the curve M at x_1. Similar results apply to the points $x_2, \cdots, x_l$. Thus the "limit" of the sequence f_j is not the curve f_∞, but rather a map $f_0\colon M \cup \bigcup_{\alpha=1}^l S^2 \to (V, J_0)$ from M with a two-sphere attached at each x_α, $\alpha = 1, \cdots, l$, to V. We denote $M \cup \bigcup_{\alpha=1}^l S^2$ by M_0 and define f_0 on M_0 by sending $x \in M - \{x_1 \cdots x_l\}$ to $f_\infty(x)$ and $x \in S_\alpha^2$ (the two-sphere attached to x_α) to $f_{\infty,\alpha}(x)$. The formation of the pseudo-holomorphic two-sphere "bubbles" is called "bubbling".

Turning to Question 2 it is clear at the outset that the "bubbling" phenomenon can occur. Moreover, since we are now allowing the complex structure on M to vary, consideration of the moduli space of conformal structures on a Riemann surface should be necessary. To do this we follow Pansu [9, §§1 and 2] in enlarging the space of pseudo-holomorphic curves (also see Gromov [6, §1.5]).

Definition. A *cusp-curve* in (V, J) is a disjoint union of Riemann surfaces M_α, together with an identification of a finite number of points (called *cuspidal points*) and a pseudo-holomorphic map $f\colon \bigcup_\alpha M_\alpha \to V$ compatible with the identifications.

Definition (*The C^k topology on cusp-curves*). Let $f\colon \bigcup_\alpha M_\alpha \to V$ be a cusp-curve. For $\varepsilon > 0$, a hermitian metric ν on M_α and a neighborhood U of the cuspidal points of f, the set of cusp-curves $\tilde{f}\colon \bigcup_\alpha \widetilde{M}_\alpha \to V$ is a neighborhood of the cusp-curve f if the following hold:

(i) There is a continuous map

$$\sigma\colon \bigcup_\alpha \widetilde{M}_\alpha \to \bigcup_\alpha M_\alpha,$$

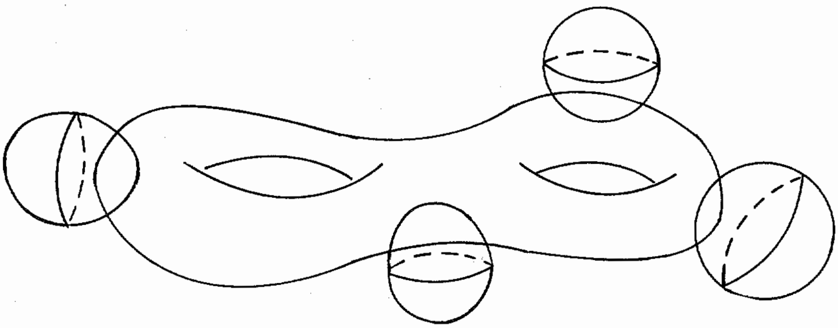
a diffeomorphism except at the cuspidal points, such that, for x cuspidal, $\sigma^{-1}(x)$ is either a simple closed curve or a cuspidal point.

(ii) $\|f - \tilde{f}\circ\sigma^{-1}\|_{C^k} < \varepsilon$ outside of U where the norm is determined by the metric ν on M_α and the given metric μ on V.

(iii) $\|J_{M_\alpha} - \sigma_* J_{\widetilde{M}_\alpha}\|_{C^k} < \varepsilon$ outside of U where the norm is as above.

(iv) $|\text{area}(f) - \text{area}(\tilde{f})| < \varepsilon$.

It is clear that a special class of cusp-curve arises from the "bubbling" phenomena. Such a curve is an arbitrary Riemann surface M with a finite number, say k, of two spheres attached at k distinct points and a pseudo-holomorphic map $f\colon M \cup \bigcup_{\alpha=1}^k S^2 \to V$.



We will call such curves *simple cusp-curves*. From Gromov's point of view cusp-curves arise by collapsing simple closed curves γ_i in a Riemann surface to points x_i. Simple cusp-curves, in this view, arise when the curves γ_i are homotopically trivial. Thus for Gromov bubbling corresponds to the collapsing of homotopically trivial simple closed curves. We have (compare [9, Theorem 5] and [6, Theorem 1.5.B]).

Theorem 1.2 (*Compactness of cusp-curves.* I). *Let (V,μ) be a compact Riemannian manifold equipped with a sequence of almost complex structures J_j uniformly bounded in $C^{k+\alpha}$. If $f_j\colon (M, J^M) \to (V, J_j)$ is a sequence of C^k pseudo-holomorphic curves of uniformly bounded μ-area, then there is a* *subsequence which converges in C^k to a simple cusp-curve $f_0\colon \bigcup_\alpha M_\alpha \to (V, J_0)$. Topologically $\bigcup_\alpha M_\alpha = M$ with a finite number of homotopically trivial simple closed curves collapsed to points.*

If the complex structure on M is permitted to vary then in the limit the curve can approach the boundary of moduli space. This means that certain simple closed curves on M which are homotopically nontrivial are collapsed to points. We have

Theorem 1.3 (*Compactness of cusp-curves.* II). *Let (V, μ) be a compact Riemannian manifold equipped with a sequence of almost complex structures J_j uniformly bounded in $C^{k+\alpha}$. Let M be a Riemann surface and J_j^M be a sequence of complex (or conformal) structures on M. If $f_j\colon (M, J_j^M) \to (V, J_j)$ is a sequence of C^k pseudo-holomorphic curves of bounded μ-area, then there is a subsequence which converges in C^k to a cusp-curve $f_0\colon \bigcup_\alpha M_\alpha \to (V, J_0)$. Topologically $\bigcup_\alpha M_\alpha = M$ with a finite number of simple closed curves collapsed to points, and so*

$$\sum_\alpha g(M_\alpha) \le g(M),$$

where g denotes the genus.

We remark that in [9] and [6] versions of Theorems 1.2 and 1.3 for pseudo-holomorphic curves with boundary are formulated and proved. The techniques we will develop to prove the above theorems apply, without difficulty, to the versions with boundary. For the sake of brevity, we leave the details to the reader. In §§2–5 we will prove Theorems 1.1 and 1.2. In §6 we show how to modify these techniques to prove Theorem 1.3.

Gromov calls the results of Theorem 1.2 and 1.3 *weak compactness for pseudo-holomorphic curves.* To obtain C^k compactness results for pseudo-holomorphic curves it is necessary to consider families of curves satisfying certain topological and geometric conditions. These conditions eliminate the possibility of cusp-curves developing and thus the weak compactness of Theorem 1.2 or 1.3 implies the C^k compactness of the family in the usual sense (see, for example, Proposition 7.5 and 7.11).

2. Some related problems and an outline of the proof

The appearance of "bubbles" in compactness problems in geometry is by now a well-known phenomenon. It's importance was first recognized by Sacks and Uhlenbeck [10]. In recent years a number of different geometric objects (e.g. minimal surfaces, harmonic maps) have been studied using a uniform bound on a related geometric functional. In the following chart we list some of these objects, the names of some workers and the geometric functional.

Geometric Object	Worker(s)	Functional
(a) pseudo-holomorphic curves	Gromov [6]	area
(b) harmonic maps of surfaces	Sacks-Uhlenbeck [10]	energy
(c) minimal surfaces	Choi-Schoen [3]	L^2 norm of the second fundamental form
(d) minimal submanifolds	M. Anderson [1]	L^n norm of the second fundmanetal form
(e) Yang-Mills connections	Uhlenbeck [15]	L^2 norm of the curvature

In all of the above works a compactness theorem is proved under the assumption of a uniform bound on the geometric functional. The phenomenon of "bubbling" either appears or can be eliminated by careful study. In any case it is a phenomenon which must be dealt with.

Another interesting feature is that the proofs of compactness in the works (b)–(e) have many features in common. Allowing for some deviations these proofs contain the following three key steps:

(1) The derivation of a formula for the Laplacian of the functional integrand. In (b) the formula is due to Eells-Sampson [4], in (c) and (d) the formula is due to J. Simons [12] and is known as Simons' equation, and in (e) the formula is due to Bourguigon-Lawson [2].

(2) Combining the formula of (1) with Morrey's mean value theorem [8] to derive uniform C^1 estimates under the assumption of suitable smallness of the functional.

(3) The use of a covering (or patching) argument to "put the pieces" of step (2) together. This argument is due to Sacks-Uhlenbeck [10] and now bears their name.

It is the purpose of Part I to provide a proof of Gromov's compactness of pseudo-holomorphic curves by following the above three steps. In this way the above problems (a)–(e) can be seen from a unified point of view.

3. A Bocher-type formula

Suppose that (V, J) is an almost complex manifold. Choose a metric μ on V, hermitian with respect to J. The triple (V, J, μ) is usually called an almost hermitian manifold. We begin by studying the geometry of (V, J, μ).

In $T^{(1,0)}V$, the cotangent bundle of $(1,0)$ forms with respect to J, choose a unitary coframe $\{\omega_1 \cdots \omega_n\}$. The first structure equation is

$$d\omega_\alpha = \omega_{\alpha\bar\beta} \wedge \omega_\beta + \theta_\alpha, \tag{3.1}$$

where $(\omega_{\alpha\bar\beta})$ is the connection one-form and θ_α is the torsion 2-form. $(\omega_{\alpha\bar\beta})$ has values in $u(n)$, that is,

$$\bar\omega_{\alpha\bar\beta} = -\omega_{\beta\bar\alpha}. \tag{3.2}$$

As θ_α is a 2-form we can write it as a sum of three 2-forms,

$$\theta_\alpha = \theta_\alpha^{(2,0)} + \theta_\alpha^{(1,1)} + \theta_\alpha^{(0,2)}, \tag{3.3}$$

where $\theta_\alpha^{(2,0)}, \theta_\alpha^{(1,1)}$ and $\theta_\alpha^{(0,2)}$ are (2,0), (1,1) and (0,2) forms respectively. Recall that $\theta_\alpha^{(0,2)} = 0$ are the necessary and sufficient conditions for the integrability of J. Write

$$\theta_\alpha^{(1,1)} = \sum_{\beta,\gamma} s_{\alpha\beta\bar\gamma}\bar\omega_\beta \wedge \omega_\gamma. \tag{3.4}$$

Then set

$$\tilde\omega_{\alpha\bar\beta} = \omega_{\alpha\bar\beta} + \sum_\gamma s_{\alpha\gamma\bar\beta}\bar\omega_\gamma - \sum_\gamma \bar s_{\beta\gamma\bar\alpha}\omega_\gamma.$$

It follows that

$$\bar{\tilde\omega}_{\alpha\bar\beta} = -\tilde\omega_{\beta\bar\alpha} \tag{3.5}$$

and that

$$d\omega_\alpha = \tilde\omega_{\alpha\bar\beta} \wedge \omega_\beta + \tilde\theta_\alpha, \tag{3.6}$$

where $\tilde\theta_\alpha^{(1,1)} = 0$. This shows that we can choose a connection compatible with J and μ whose (1,1) torsion vanishes. It is easy to verify that this connection is unique. We will call this connection *compatible* and denote it by $\omega_{\alpha\bar\beta}$ (i.e. we will henceforth drop the tilde). The construction of the compatible connection follows the standard procedure in the theory of G-structures of "absorbing the torsion into the connection". Set

$$d\omega_{\alpha\bar\beta} - \omega_{\alpha\bar\gamma} \wedge \omega_{\gamma\bar\beta} = \Omega_{\alpha\bar\beta}. \tag{3.7}$$

$\Omega_{\alpha\bar\beta}$, the curvature of the compatible connection, is a 2-form with values in $u(n)$. Denote by $\Omega_{\alpha\bar\beta}^{(1,1)}$ the (1,1) part of $\Omega_{\alpha\bar\beta}$ and write

$$\Omega_{\alpha\bar\beta}^{(1,1)} = \sum_{\gamma,\delta} R_{\alpha\bar\beta\gamma\bar\delta}\bar\omega_\gamma \wedge \omega_\delta. \tag{3.8}$$

Let $f\colon (M, J^M) \to (V, J)$ be a pseudo-holomorphic curve. Let ϕ be a (1,0) form on M for the complex structure J^M. The condition that f is pseudo-holomorphic can be expressed as

$$f^*\omega_\alpha = a_\alpha \phi, \qquad \alpha = 1, \cdots, n, \tag{3.9}$$

where the a_α are complex valued functions on M. The energy of f is, by definition,

$$E(f) = \int_M \sum |a_\alpha|^2 \phi \wedge \bar{\phi}. \tag{3.10}$$

Note that $E(f)$ depends only on the conformal class (or complex structure) of M. The area of the curve f with respect to the metric μ is

$$\text{area}(M, f^*\mu) = \int_M \sum |a_\alpha|^2 \phi \wedge \bar{\phi}. \tag{3.11}$$

It follows that area bounds give energy bounds. Moreover, energy is a more convenient object to work with since it is conformally invariant in the domain manifold. We will call the function $\sum |a_\alpha|^2$ the *energy integrand.* Of course this function depends on a choice of metric on M in the conformal class defined by ϕ. At this time we make such a choice and call this metric ν.

Set

$$e(f) = \sum |a_\alpha|^2. \tag{3.12}$$

Step (1) of the outline of the proof in §2 indicates that we must compute the quantity $\Delta e(f)$, where Δ is the Laplace-Beltrami operator on (M, ν). To do this we return to (3.9), and take the exterior derivative

$$d\omega_\alpha = (da_\alpha - i a_\alpha \rho) \wedge \phi, \tag{3.13}$$

where ρ is the connection 1-form of ν. By (3.6) the torsion form θ_α is a sum of forms of type (2,0) and (0,2) and so θ_α vanishes on M. It follows that

$$d\omega_\alpha = \omega_{\alpha\bar{\beta}} \wedge a_\beta \phi. \tag{3.14}$$

(3.13) and (3.14) yield

$$(da_\alpha - i a_\alpha \rho - a_\beta \omega_{\alpha\bar{\beta}}) \wedge \phi = 0. \tag{3.15}$$

By Cartan's Lemma we have

$$Da_\alpha \equiv da_\alpha - i a_\alpha \rho - a_\beta \omega_{\alpha\bar{\beta}} = b_\alpha \phi. \tag{3.16}$$

Taking the exterior derivative of (3.16) and using (3.7),

$$(db_\alpha - 2i\rho b_\alpha - b_\beta \omega_{\alpha\bar{\beta}}) \wedge \phi = -i a_\alpha d\rho - a_\gamma \Omega_{\alpha\bar{\gamma}}. \tag{3.17}$$

The covariant derivative of b_α is

$$Db_\alpha = db_\alpha - 2i\rho b_\alpha - b_\beta \omega_{\alpha\bar{\beta}}. \tag{3.18}$$

The Gauss curvature K of the metric ν satisfies

$$d\rho = -i(K/2)\phi \wedge \bar{\phi}. \tag{3.19}$$

Using (3.8), (3.18) and (3.19), (3.17) becomes

$$Db_\alpha \wedge \phi = -(K/2)a_\alpha \phi \wedge \bar{\phi} + a_\gamma \bar{a}_\beta a_\delta R_{\alpha\bar{\gamma}\beta\bar{\delta}} \phi \wedge \bar{\phi}. \tag{3.20}$$

Thus

$$\Delta a_\alpha = (K/2)a_\alpha - a_\gamma \bar{a}_\beta a_\delta R_{\alpha\bar{\gamma}\beta\bar{\delta}}, \tag{3.21}$$

and hence

$$\begin{aligned} \frac{1}{2}\Delta\left(\sum_\alpha |a_\alpha|^2\right) = \sum_\alpha |\nabla a_\alpha|^2 + \frac{1}{2}K\left(\sum_\alpha |a_\alpha|^2\right) \\ - \sum \bar{a}_\alpha a_\gamma \bar{a}_\beta a_\delta R_{\alpha\bar{\gamma}\beta\bar{\delta}}. \end{aligned} \tag{3.22}$$

It follows that there exist positive constants C_1 and C_2 where C_1 depends on ν and C_2 depends on μ and J such that

$$\Delta e(f) \geq -C_1 e(f) - C_2(e(f))^2, \tag{3.23}$$

which is the required formula.

4. C^k estimates

The following theorem and its proof are modelled on analogous work on harmonic maps due to R. Schoen [11].

Theorem 4.1. *Let (V, J, μ) be an almost hermitian manifold and let ν be a hermitian metric on $D_r = \{x \in \mathbb{C}\colon |x| < r\}$ which satisfies $L^{-1}(\delta_{ij}) \leq \nu \leq L(\delta_{ij})$. If $f\colon D_r \to V$ is a C^1 pseudo-holomorphic curve then there exists an $\varepsilon_0 > 0$ such that if $\int_{D_r} e(f) \leq \varepsilon_0$, then f satisfies*

$$\max_{\sigma\in(0,r]} \sigma^2 \sup_{D_{r-\sigma}} e(f) \leq C_0. \tag{4.1}$$

In particular,

$$\sup_{D_{r/2}} e(f) \leq \frac{4C_0}{r^2}. \tag{4.2}$$

ε_0 and C_0 depend on the geometry of V and on L but are independent of f.

Proof. Let $\sigma_0 \in (0, r]$ be chosen such that

$$\sigma_0^2 \sup_{D_{r-\sigma_0}} e(f) = \max_{\sigma\in(0,r]} \sigma^2 \sup_{D_{r-\sigma}} e(f).$$

Let $x_0 \in \overline{D}_{r-\sigma_0}$ be chosen so that

$$e(f)(x_0) = \sup_{D_{r-\sigma_0}} e(f),$$

and hence

$$\sup_{D_{\sigma_0/2}(x_0)} e(f) \le 4e(f)(x_0). \tag{4.3}$$

If $\sigma_0^2 e(f)(x_0) \le 4$ then (4.1) is true, so we may assume that

$$e(f)(x_0) \ge 4\sigma_0^{-2}. \tag{4.4}$$

Set $e_0 = e(f)(x_0)$ and $r_0 = e_0^{1/2}\sigma_0/2$. Define a mapping $g\colon D_{r_0} \to V$ by $g(y) = f((y - x_0)/e_0^{1/2})$. g is a rescaled version of f chosen so that $e(g)(0) = 1$, $\sup_{D_{r_0}} e(g) \le 4$ (by (4.3)), $r_0 \ge 1$ (by (4.4)). Therefore from (3.23) we have on D_{r_0}

$$\Delta e(g) \ge C_3 e(g). \tag{4.5}$$

Applying the mean value theorem of Morrey [8] (or [5, Theorem 9.20]) on D_1 we have

$$1 = e(g)(0) \le C_4 \int_{D_1} e(g). \tag{4.6}$$

But

$$\int_{D_1} e(g) = \int_{D_{e_0^{-1/2}}(x_0)} e(f) \le \int_{D_r} e(f) \le \varepsilon_0. \tag{4.7}$$

By choosing ε_0 sufficiently small (depending on C_4) we get a contradiction from (4.6) and (4.7). q.e.d.

Suppose now that $\{J_j\}$ is a sequence of almost complex structures uniformly bounded in $C^{k+\alpha}$, $k \ge 2$, and suppose that $f_j\colon (M, J^M) \to (V, J_j)$ is a sequence of C^k pseudo-holomorphic curves. For each j we choose a metric μ_j on V, hermitian with respect to J_j, such that

$$\|\mu_j\|_{C^2} \le B \quad \text{and} \quad \|\mu_j^{-1}\|_{C^0} \le B, \tag{4.8}$$

where the norm $\|\ \|$ is taken using the given metric μ. The curvatures of the μ_j are, then, uniformly bounded in C^0.

Recall that the energy integrand depends on both the domain and target metrics. We will denote the energy of f_j with respect to μ_j by $e_{\mu_j}(f_j)$. The metric ν on M is fixed.

Corollary 4.2. *Let ν and D_r be as in Theorem 4.1, and suppose that $f_j\colon D_r \to (V, J_j, \mu_j)$ is a sequence of pseudo-holomorphic curves, where the J_j are bounded in $C^{k+\alpha}$, $k \ge 2$, and the μ_j are hermitian metrics adapted to* *the J_j as above. Then, for all $j = 1, 2, \cdots$, there exists an $\varepsilon_0 > 0$ such that if $\int_{D_r} e_{\mu_j}(f_j) \leq \varepsilon_0$ then f_j satisfies*

$$\sup_{D_{r/2}} e_{\mu_j}(f_j) \leq 4C_0/r^2, \tag{4.9}$$

where ε_0 and C_0 are independent of j.

Proof. Using the uniform C^0 bounds on the curvatures of the μ_j the constant C_2 of (3.23) can be chosen uniformly. It follows that the constants C_3 and C_4 of (4.5) and (4.6) are independent of j and so ε_0 and C_0 are also independent of j.

Remark. Theorem 4.1 and Corollary 4.2 provide C^1 estimates under an assumption of small energy. The pseudo-holomorphic map equations are a first order nonlinear elliptic system with coefficients satisfying uniform $C^{k+\alpha}$ bounds. Using classical Schauder theory the C^1 estimates (4.9) can be bootstrapped up to give uniform C^k estimates for the maps f_j,

$$\|f_j\|_{C^k(\mu_j)} \leq C_5. \tag{4.10}$$

Here C_5 is independent of j, but the C^k norm of f_j depends on the metric μ_j. The metrics μ_j and their inverses μ_j^{-1} are uniformly bounded in C^0. In particular, there is a constant λ such that for all j we have

$$\lambda^{-1}\mu \leq \mu_j \leq \lambda\mu. \tag{4.11}$$

It follows that there is a constant C depending on C_0 and λ and an $\varepsilon > 0$ depending on ε_0 and λ such that, for all f_j, if

$$\int_{D_r} e_\mu(f_j) < \varepsilon, \tag{4.12}$$

then

$$\|f_j\|_{C^k(\mu)} < C. \tag{4.13}$$

This is the required C^k estimate.

5. The Sacks-Uhlenbeck covering argument and bubbling

We begin by proving Theorem 1.1.

Lemma 5.1. *Under the hypotheses of Theorem* 1.1 *the condition* $\text{area}_\mu(f_j) \leq A$ *implies that there is a constant A' independent of j such that*

$$\text{energy}_\mu(f_j) = \int_M e_\mu(f_j) \leq A'. \tag{5.1}$$

Proof. Choose metrics μ_j, hermitian with respect to J_j, as in §4. In particular there is a constant λ, independent of j, such that for all j

$$\lambda^{-1}\mu < \mu_j < \lambda\mu. \tag{5.2}$$

We observed in §3 that

$$\operatorname{area}_{\mu_j}(f_j) = \operatorname{energy}_{\mu_j}(f_j). \tag{5.3}$$

Using this and inequality (5.2) twice yields the result.

Proof of Theorem 1.1. Choose $r_0 > 0$ and set $r_m = 2^{-m}r_0$, $m \in \mathbb{Z}_+$. For each m take a finite covering $\mathscr{C}_m = \{D_{r_m}(y_\alpha)\}$ of M such that each point of M is covered at most h times by discs in $\mathscr{C}_m$, where h depends on M only, and such that $\{D_{r_{m/2}}(y_\alpha)\}$ is still a covering of M. For each j,

$$\sum_\alpha \int_{D_{r_m}(y_\alpha)} e_\mu(f_j) \leq hA'.$$

Thus for each j there are at most hA'/ε discs on which

$$\int_{D_{r_m}(y_\alpha)} e_\mu(f_j) \geq \varepsilon.$$

The center points of these discs make at most hA'/ε sequences of points of M (by letting $j = 1, 2, \cdots$). Since $\mathscr{C}_m$ is a finite covering and M is compact we may assume these center points are fixed by passing to a subsequence of $\{f_j\}$ (which we will continue to denote $\{f_j\}$). For each m, call these center points $\{x_{1,m}, \cdots, x_{l,m}\}$ where l is an integer and $l \leq hA'/\varepsilon + 1$. By our C^k estimates and the Ascoli theorem we can successively choose a subsequence of $\{f_j\}$ which converges (in C^k) in every disc $D_{r_m/2}(y_\alpha)$ for each $D_{r_m}(y_\alpha) \in \mathscr{C}_m$ except for at most l discs of $\mathscr{C}_m$. Let $m \to \infty$. We can choose a subsequence of $\{m\}$ such that $\{x_{1,m}\}, \cdots, \{x_{l,m}\}$ converge to points $x_1, \cdots, x_l$. Choosing a diagonal subsequence of $\{f_j\}$ finishes the proof.

Remark. The above proof is due essentially to Sacks-Uhlenbeck [10]. The version given here was adapted from a similar argument in [3].

It remains to determine what happens at the points x_s, $s = 1, \cdots, l$. This is where the phenomenon of bubbling occurs. We will need the following:

Removable Singularity Theorem. *Let (V, μ) be a compact Riemannian manifold equipped with an almost complex structure J of class $C^{k+\alpha}$, $k \geq 1$. Let $f\colon D - \{0\} \to V$ be a C^k pseudo-holomorphic map. If f has finite area, then f can be extended to a C^k pseudo-holomorphic map $D \to V$.*

For a proof of this result see [9].

The work of Sacks and Uhlenbeck applies directly and essentially without modification to our situation. The reader can simply read Lemma 4.5 and

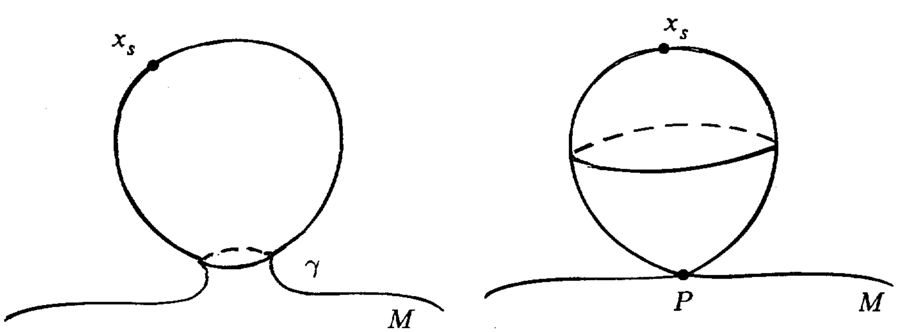
Theorem 4.6 of [10] and substitute "pseudo-holomorphic" for "harmonic". However for completeness we briefly summarize this work.

First note that the pseudo-holomorphic curves f_j converge in C^1 to f_0 in a neighborhood of x_s, $s = 1, \cdots, l$, unless $e_\mu(f_j)(x) \to \infty$ as $j \to \infty$ and $x \to x_s$. This follows easily from our C^1 estimates. Assume now that at x_s, C^1 convergence fails. For some sufficiently small $\delta > 0$ we set

$$b_j = \max_{x \in D_\delta(x_s)} e_\mu(f_j),$$

and let x_j be the point in $D_\delta(x_s)$ where this maximum is attained. Then $b_j \to \infty$ and $x_j \to x_s$ as $j \to \infty$. Set $\tilde{f}_j = f_j(x_j + x/b_j)$. Then $\tilde{f}_j \colon D_{\delta b_j}(0) \to (V, J_j)$ is a pseudo-holomorphic curve, $e_\mu(\tilde{f}_j)(x) \leq 1$ for $x \in D_{\delta b_j}(0)$, and $e_\mu(\tilde{f}_j)(0) = 1$. The discs on which the $\tilde{f}_j$ are defined have radii going to ∞ as $j \to \infty$ and consequently the metrics on these discs converge to the euclidean metric. By Theorem 1.1 for any $R > 0$ we can find a subsequence of the $\tilde{f}_j$ which converge in C^k to a pseudo-holomorphic curve $\tilde{f}_0 \colon D_R(0) \to (V, J_0)$. This convergence is C^k on all of $D_R(0)$ since by the choice of the subsequence we can suppose $e_\mu(\tilde{f}_j) \leq 1$ on $D_R(0)$. Since $e_\mu(\tilde{f}_0)(0) = 1$, $\tilde{f}_0$ cannot be a map to a point. Since R is arbitrary we get a pseudo-holomorphic map $\tilde{f}_0$ from $\mathbb{R}^2$ into (V, J_0). But $\mathbb{R}^2 = S^2 - \{p\}$ conformally, so $\tilde{f}_0 \colon S^2 - \{p\} \to (V, J_0)$ is pseudo-holomorphic. Using the removable singularity theorem we get a pseudo-holomorphic curve $\tilde{f}_0 \colon S^2 \to (V, J_0)$. Thus at each point x_s where C^1 convergence fails a pseudo-holomorphic curve of genus zero "bubbles" off.

Recall the description of bubbling given in §2 as the collapsing of a homotopically trivial simple closed curve γ in M to a point. In the above notation the curve γ is $\partial D_{\delta b_j} - 1(x_j)$. As $j \to \infty$ the curve γ converges to the point $p \in S^2$, the south pole; while x_s becomes the north pole.



This is bubbling.

6. The case of varying conformal structure

Suppose now that the sequence f_j are C^k pseudo-holomorphic maps $(M, J_j^M) \to (V, J_j)$ where the J_j are bounded as above, the J_j^M are a sequence of complex structures on M and genus $(M) \geq 2$. In each conformal class determined by J_j^M choose the metric ν_j of constant curvature -1. If as $j \to \infty$ the Riemann surfaces (M, J_j^M, ν_j) approach the boundary of moduli space, then on M we can find homotopically nontrivial simple closed curves $\gamma_1, \cdots, \gamma_t$ which are geodesics for the metrics ν_j and whose ν_j lengths go to zero as $j \to \infty$. Denote $M - \bigcup_{s=1}^t \gamma_s$ by $\widetilde{M}$ and let $X \subset \widetilde{M}$ be a compact subset. On X the hyperbolic metrics ν_j converge in C^∞ to a smooth hyperbolic metric ν_∞. Denote by Δ_{ν_j} the Laplace-Beltrami operator of the metric ν_j. On X the operators Δ_{ν_j} are strictly elliptic and their coefficients are uniformly bounded. To utilize the results of §3 we must compute the energy integrand of f_j using the metrics ν_j and μ_j. Denote this by $e_{\nu_j,\mu_j}(f_j)$. Then by (3.23)

$$\Delta e_{\nu_j,\mu_j}(f_j) \geq -C_1(e_{\nu_j,\mu_j}(f_j)) - C_2(e_{\nu_j,\mu_j}(f_j))^2, \tag{6.1}$$

where C_1 is a constant independent of j (since the metrics ν_j are all of constant curvature -1) and C_2 is a constant also independent of j (as shown in §4). Tracing through the proof of Theorem 4.1, the constant C_3 can be taken independent of j. Since the coefficients of Δ_{ν_j} are uniformly bounded, Morrey's theorem implies that the constant C_4 can be taken to be independent of j. It follows that there is an $\varepsilon_0 > 0$, independent of j, such that if $D_r \subset X$ and

$$\int_{D_r} e_{\nu_j,\mu_j}(f_j) \leq \varepsilon_0, \tag{6.2}$$

then

$$\sup_{D_{r/2}} e_{\nu_j,\mu_j}(f_j) \leq \frac{4C_0}{r^2}, \tag{6.3}$$

where C_0 is a constant independent of j and the radius r is computed using the metric ν_j. Using Schauder theory, as in §4, this C^1 estimate can be bootstrapped to yield uniform C^k estimates for the maps f_j under the "small energy assumption" (6.2),

$$\|f_j\|_{C^k(\nu_j,\mu_j)} \leq C_5. \tag{6.4}$$

C_5 is independent of j. On X the metrics ν_j converge in C^∞ to ν_∞ so by taking a subsequence, if necessary, there is a constant τ independent of j so that

$$\tau^{-1}\nu_\infty < \nu_j < \tau\nu_\infty. \tag{6.5}$$

(6.5) and (4.11) together imply that there is a constant C depending on C_0, λ and τ and an $\varepsilon > 0$ depending on ε_0, λ and τ such that, for all f_j, if $D_r \subset X$ and

$$\int_{D_r} e_{\nu_\infty,\mu}(f_j) < \varepsilon, \tag{6.6}$$

then

$$\|f_j\|_{C^k(\nu_\infty,\mu)} < C, \tag{6.7}$$

where r is computed with respect to the metric ν_∞. The argument of §5 now completes the proof of Theorem 1.3 when genus $(M) \geq 2$. We have the case where genus $(M) = 1$ to the reader.

PART II
APPLICATIONS TO SYMPLECTIC GEOMETRY

In this part we outline the proof of one of Gromov's theorems in symplectic geometry. In §7 we first show how Gromov uses the taming condition to provide area bounds. These bounds, the J-simplicity condition (a homotopy condition) and the results of Part I then yield compactness results. In §8 the compactness results are framed in the language of infinite dimensional manifolds. Finally the main theorem is proved in §9.

7. Taming, J-simple curves and compactness

Let (N, ω) be a compact symplectic manifold of dimension $2n$ and let μ be a Riemannian metric on N. The symplectic form ω is, by definition, a closed nondegenerate 2-form on N.

Definition 7.1. An almost complex structure J on N is said to be *tamed* by ω if $\omega(v, Jv) > 0$ for each $x \in N$ and each $v \in T_xN$. That is, J is tamed by ω if ω is positive on each J-complex line of T_xN for all $x \in N$.

Suppose J is an almost complex structure on N tamed by ω.

Proposition 7.2. *Let M be a Riemann surface and $f\colon M \to N$ be a J-holomorphic curve which represents a homology class $\beta \in H_2(N;\mathbb{Z})$. Then* $\text{area}_\mu(f) \leq A$, *where A is a constant depending on ω, β, μ and $\|J\|_{C^0}$ but is independent of f.*

Proof. Using the compactness of N we can suppose that the metric μ is hermitian with respect to J and we can find a constant c depending on μ and $\|J\|_{C^0}$ so that

$$\omega(v, Jv) \geq c^{-1}\mu(v, v)$$

for all $v \in T_xN$ and $x \in N$. Thus

$$\text{area}_{\mu/c}(f) \leq \int_M f^*(\omega) = [\omega](\beta),$$

where $[\omega]$ denotes the cohomology class represented by ω. The result follows.

Definition 7.3 [6, 1.5.E$_2$]. A homotopy class β of maps $S^2 \to (N, J)$ is called *J-simple* if every J-holomorphic cusp-curve in β,

$$f\colon \bigcup_{i=1}^{l} S_i^2 \to N,$$

is nonconstant on at most one of the two-spheres, S_i^2, $i = 1, \cdots, l$.

Example 7.4. A homotopy class β is J-simple if it admits no decomposition

$$\beta = \sum_{i=1}^{l} \beta_i \quad \text{for } l \geq 2,$$

where each β_i can be realized by a nonconstant rational J-holomorphic curve.

For $k \geq 1$, let

$$\mathscr{J}_B = \{J\colon J \text{ is an almost complex structure on } N \text{ tamed by } \omega \text{ and satisfies } \|J\|_{C^{\alpha+k}} \leq B\}.$$

Let β be a homotopy class in N, which is J-simple for each $J \in \mathscr{J}_B$. Choose three distinct points $s_i \in S^2$, $i = 1, 2, 3$, and $\delta > 0$.

Proposition 7.5 [6, 1.5.E$'_2$]. *The space of C^k maps $f\colon S^2 \to N$ in the homotopy class β and satisfying*

(i) *f is J-holomorphic for some $J \in \mathscr{J}_B$, and*

(ii) $\text{dist}(f(s_i), f(s_j)) \geq \delta$ *for $i \neq j$,*

is compact.

Remark 7.6. Condition (ii) is necessary. If f is J-holomorphic and g is an element of the group of conformal self-maps of S^2, then $f \circ g$ is J-holomorphic. As the conformal self-maps of S^2 form a noncompact group, the set of J-holomorphic maps $S^2 \to N$, $\{f \circ g\colon f\colon S^2 \to N \text{ is } J\text{-holomorphic and } g\colon S^2 \to S^2 \text{ is conformal}\}$, is noncompact. However if $s_i \in S^2$, $i = 1, 2, 3$, are distinct points and $\text{dist}(s_i, s_j) \geq \varepsilon$, $i \neq j$, for some $\varepsilon > 0$, then the subgroup $\{g\colon g\colon S^2 \to S^2 \text{ is conformal and } \text{dist}(g(s_i), g(s_j)) \geq \varepsilon \text{ for } i \neq j\}$ is compact.

Proof. If f satisfies the hypotheses of the proposition then by Proposition 7.2 $\text{area}_\mu(f) \leq A$, where A depends on ω, β, μ and B. Let $\{f_j\}$ be a sequence of J_j-holomorphic curves $S^2 \to N$ satisfying the hypotheses of the theorem, where $J_j \in \mathscr{J}_B$. We can extract a subsequence, which we still denote by $\{f_j\}$, such that the $\{J_j\}$ converge in $C^{k+\alpha}$ to an almost complex structure $J_\infty \in \mathscr{J}_B$

and using Theorem 1.2 such that the $\{f_j\}$ converge to a J_∞-holomorphic cusp-curve f_∞. Write

$$f_\infty\colon \bigcup_{i=1}^{l} S_i^2 \to N.$$

f_∞ represents the class β and so by the J-simplicity of β, f_∞ is nonconstant on at most one of the S_i^2, $i = 1, \cdots, l$. By (ii) f_∞ is nonconstant so we can assume that f_∞ is nonconstant only on, say, S_1^2. If S_1^2 is a "bubble" then the images of two of the s_i have converged together in the limit. This violates (ii). Hence the $\{f_j\}$ converge in C^k to a J_∞-holomorphic curve $f_\infty\colon S^2 \to N$.

Example 7.7 [6, 2.3.C]. Let $N = \mathbb{C}P^r \times V$ and $\omega = \omega_1 \oplus \omega_2$, where ω_1 is the Kähler form for the Fubini-Study metric on $\mathbb{C}P^r$, $r \geq 1$, and where the symplectic form ω_2 on V satisfies the following integrality condition:

For any smooth mapping $x\colon S^2 \to V$

$$\int_{x(S^2)} \omega_2 = k \int_{\mathbb{C}P^1} \omega_1, \tag{7.8}$$

where k is an integer depending on the homotopy class of x.

Let β be the homotopy class of maps

$$S^2 \xrightarrow{\simeq} \mathbb{C}P^1 \subset \mathbb{C}P^r \times v \qquad v \in V, \tag{7.9}$$

where $\mathbb{C}P^1 \subset \mathbb{C}P^r \times v$ is the linear embedding.

Lemma 7.10. *If ω_2 satisfies the integrality condition* (7.8), *then the homotopy class β* (7.9) *is J-simple for any almost complex structure J on N tamed by $\omega_1 \oplus \omega_2$.*

Proof. Let J be an almost complex structure on N tamed by $\omega_1 \oplus \omega_2$ and suppose that β admits a decomposition $\beta = \sum_{i=1}^{l} \beta_i$, $l \geq 2$, where each β_i can be realized by a nonconstant rational J-holomorphic curve. Set $\int_{\mathbb{C}P^1} \omega_1 = a$. For each i, using the positivity of J-holomorphic curves and (7.8),

$$0 < [\omega_1 \oplus \omega_2](\beta_i) = [\omega_1](\beta_i) + [\omega_2](\beta_i) = m_i a + n_i a,$$

where m_i and n_i are integers. Thus, for each i, $m_i + n_i$ is a positive integer. But by (7.9) $[\omega_1 \oplus \omega_2](\beta) = a$ so

$$a = \sum_{i=1}^{l} [\omega_1 \oplus \omega_2](\beta_i) = a \sum_{i=1}^{l} (m_i + n_i).$$

This is impossible if $l \geq 2$. q.e.d.

We impose a compactness condition in this example as follows: Let $v \in V$ and choose three embedded disjoint submanifolds Σ_i of N which transversally intersect $\mathbb{C}P^r \times v \subset N$ and satisfy:

(a) $\Sigma_i \cap \mathbb{C}P^r \times v$ is a single point for $i = 1, 2$,

(b) $\Sigma_3 \cap \mathbb{C}P^r \times v$ has codimension two in $\mathbb{C}P^r \times v = \mathbb{C}P^r$ and is not homologous to zero in $H_{2r-2}(\mathbb{C}P^r; \mathbb{Z}_2)$.

Choose three distinct points $s_i \in S^2$, $i = 1, 2, 3$.

Proposition 7.11. *The space of C^k maps $f \colon S^2 \to \mathbb{C}P^r \times V$ in the homotopy class β (7.9) and satisfying*

(i) *f is J-holomorphic for some J tamed by $\omega_1 \oplus \omega_2$ with $\|J\|_{C^{\alpha+k}} \leq B$,*

(ii) *$f(s_i) \in \Sigma_i$, $i = 1, 2, 3$,*

is compact.

Proof. Immediate from Proposition 7.5.

In the next two sections we will need the following

Proposition 7.12 [6, 2.3.C$'_2$]. *If (N, ω) is a symplectic manifold, not necessarily compact, then the space of almost complex structure on N tamed by ω is contractible. In particular, this space is connected.*

Remark. The space of almost complex structures on N tamed by ω is nonempty. For a proof see [16, Lecture 2].

8. The global set-up

In this section we follow, for the most part, the work of D. McDuff [7].

Let $\beta \in H_2(N; \mathbb{Z})$ and let M be a Riemann surface of genus g. Let $F = F^k_\beta$ be the set of all C^k maps, $k \geq 1$, $f \colon M \to N$ which represent the class β and are immersions except possibly at a finite number of points. Let $J = \mathscr{J}^{k+\alpha}$ be an open connected subset of the space of all $C^{k+\alpha}$ almost complex structures on N. F and $\mathscr{J}$ are Banach manifolds. Define

$$\mathscr{M}_\beta = \mathscr{M}^k_\beta = \{(f, J) \in F \times \mathscr{J} : f \text{ is } J\text{-holomorphic}\}.$$

Proposition 8.1. *$\mathscr{M}_\beta$ is a Banach manifold.*

Proof. See [7, Lemma 4.1].

Now consider the projection $P_\beta : \mathscr{M}_\beta \to \mathscr{J}$:

Proposition 8.2. *P_β is Fredholm.*

Proof. See [7, Proposition 4.2].

In fact it follows from McDuff's work that the index of dP_β is equal to the index of a first order elliptic linear differential operator.

Proposition 8.3. *The index of P_β is $2(c + n(1-g))$, where $2n = \dim_{\mathbb{R}} N$ and c is the value of the first Chern class of N, $c_1(N)$, evaluated on the homology class β.*

Proof. Apply the Atiyah-Singer index theorem. Also see [6, 2.1.A].

It follows from the Sard-Smale theorem [13] that, for a generic almost complex structure $J \in \mathscr{J}$, $P^{-1}_\beta(J)$ is a smooth manifold of dimension

$2(c + n(1 - g))$. To utilize the full power of the Sard-Smale theorem it is necessary to show that the Fredholm map P_β is proper. Properness implies, for example, that $P_\beta^{-1}(J)$ is a compact manifold. We have seen in Remark 7.6 that, in general, this is not the case. To show that P_β is proper requires a compactness theorem. How this theorem is formulated depends on the application at hand.

Example 7.7 (*continued*). Let β be the homology class which represents the image of (7.9). Let $\widetilde{F} = \widetilde{F}_\beta^k$ be the set of all C^k maps, $k \geq 1$, $f\colon S^2 \to \mathbb{C}P^r \times V$ which represent β, are immersions except possibly at a finite number of points and satisfy $f(s_i) \in \Sigma_i$, $i = 1, 2, 3$. Let $\mathscr{J} = \mathscr{J}^{k+\alpha}$ be the set of almost complex structures on N tamed by $\omega_1 \oplus \omega_2$. Define

$$\widetilde{\mathscr{M}}_\beta = \widetilde{\mathscr{M}}_\beta^k = \{(f, J) \in \widetilde{F} \times \mathscr{J} : f \text{ is } J\text{-holomorphic}\}.$$

As above $\widetilde{\mathscr{M}}_\beta^k$ is a Banach manifold and the projection $\widetilde{P}_\beta\colon \widetilde{\mathscr{M}}_\beta \to \mathscr{J}$ is Fredholm. By choosing the submanifolds Σ_i, $i = 1, 2, 3$, generically, the conditions $f(s_1) \in \Sigma_1$ and $f(s_2) = \Sigma_2$ each impose $2r$ linear conditions on $\ker d\widetilde{P}_\beta$ and the condition $f(s_3) \in \Sigma_3$ imposes two linear conditions on $\ker d\widetilde{P}_\beta$. Since $c = r + 1$, we have

$$\operatorname{index} \widetilde{P}_\beta = 2(n - r).$$

Moreover, Proposition 7.11 implies that $\widetilde{P}_\beta$ is a proper mapping.

9. An application to symplectic geometry

We continue with Example 7.7.

Theorem 9.1 [6, 2.3.C]. *Given any point $\sigma_1 \in \Sigma_1$ and any almost complex structure J on $\mathbb{C}P^r \times V$ tamed by $\omega_1 \oplus \omega_2$, there exists a J-holomorphic map $f\colon S^2 \to \mathbb{C}P^r \times V$ homotopic to $S^2 \xrightarrow{\simeq} \mathbb{C}P' \subset \mathbb{C}P^r \times v \subset \mathbb{C}P^r \times V$ such that $f(s_i) \in \Sigma_i$, $i = 1, 2, 3$, and $f(s_1) = \sigma_1$.*

Proof. Let J' denote the Kähler complex structure on $\mathbb{C}P^r$ and J'' denote any almost complex structure on V tamed by ω_2. Then $J_0 = J' \oplus J''$ is an almost complex structure tamed by $\omega_1 \oplus \omega_2$. Let $l\colon S^2 \xrightarrow{\simeq} \mathbb{C}P^1 \subset \mathbb{C}P^r = \mathbb{C}P^r \times v \subset \mathbb{C}P^r \times V$ be the linear embedding. We remark that there is a unique linear $\mathbb{C}P^1 \to \mathbb{C}P^r$ which passes through the two points $\Sigma_1 \cap \mathbb{C}P^r \times v$ and $\Sigma_2 \cap \mathbb{C}P^r \times v$. Because $\Sigma_3 \cap \mathbb{C}P^r \times v$ is not homologous to zero in $H_{2r-2}(\mathbb{C}P^r; \mathbb{Z}_2)$ for generic choice of the Σ_i's this line also intersects Σ_3 in a single point. Now choose $s_i \in S^2$, $i = 1, 2, 3$, such that $l(s_i) = \Sigma_i \cap \mathbb{C}P^r \times v$. This shows that $\widetilde{P}_\beta^{-1}(J_0)$ is not empty.

It is a classical computation that the tangent space to the space of complex curves in $\mathbb{C}P^r$ at l can be identified with $H^0(\mathbb{C}P^1; T_*\mathbb{C}P^r/T_*\mathbb{C}P^1)$ and this space has complex dimension $2(r-1)$. It follows that the tangent space to the space of J_0-holomorphic curves in $\mathbb{C}P^r \times V$ at l has real dimension $4(r-1)+2(n-r)$. (The manifold V of real dimension $2(n-r)$ plays the role of a parameter space.) If the Σ_i, $i=1,2,3$, are chosen generically then requiring that the (unparametrized) J_0-holomorphic curves intersect Σ_1 and Σ_2 imposes $4(r-1)$ linear conditions on the tangent space. The condition that the curves intersect Σ_3 holds automatically. Hence $\dim_{\mathbb{R}} \ker(d\widetilde{P}_\beta)_{(l,J_0)} = 2(n-r)$. Comparing this with Proposition 8.3 shows that (l, J_0) is a regular point and J_0 is a regular value of $\widetilde{P}_\beta$ in the sense of Sard-Smale. We can conclude, in particular, that $\widetilde{P}_\beta^{-1}(J_0)$ is a nonempty compact manifold of real dimension $2(n-r)$.

Let J_1 be any generic (i.e., regular, in the sense of Sard-Smale) almost complex structure on $\mathbb{C}P^r \times V$ tamed by $\omega_1 \oplus \omega_2$. Since J_0 is also generic there is a path $\gamma\colon [0,1] \to \mathscr{J}$ joining J_0 to J_1 which is transverse to $\widetilde{P}_\beta$. It follows that $\widetilde{P}_\beta^{-1}(\gamma[0,1])$ is a submanifold of $\widetilde{F}$ of dimension $2(n-r)+1$. Under the evaluation map $e\colon \widetilde{F} \to \Sigma_1$, $e(f) = f(s_1)$, the homology class $[\widetilde{P}_\beta^{-1}(J_0)] \in H_{2(n-r)}(\widetilde{F}; \mathbb{Z}_2)$ goes to $[\Sigma_1] \in H_{2(n-r)}(\Sigma_1; \mathbb{Z}_2)$. As $\widetilde{P}_\beta^{-1}(J_1)$ is cobordant to $\widetilde{P}_\beta^{-1}(J_0)$ the homology class $[P_\beta^{-1}(J_1)] \in H_{2(n-r)}(\widetilde{F}; \mathbb{Z}_2)$ also goes to $[\Sigma_1]$ under e. But this implies that $e\colon \widetilde{P}_\beta^{-1}(J_1) \to \Sigma_1$ is onto. Hence there is a J_1-holomorphic map $f\colon S^2 \to \mathbb{C}P^r \times V$ satisfying $f(s_1) = \sigma_1$, $f(s_i) \in \Sigma_i$, $i = 1,2,3$, and f is homotopic to the linear embedding $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^r$.

Now let J be any almost complex structure on $\mathbb{C}P^r \times V$ tamed by $\omega_1 \oplus \omega_2$. Choose a sequence $\{J_j\}$ of generic almost complex structures which converge to J in $C^{k+\alpha}$. For each j let f_j be a J_j-holomorphic map $S^2 \to \mathbb{C}P^r \times V$ constructed above. The simplicity of the class β implies that a subsequence of the $\{f_j\}$ converges to the required map.

Lemma 9.2. *Let (N, ω) be a compact symplectic manifold and let $U \subset N$ be an open submanifold equipped with a $C^{k+\alpha}$ almost complex structure J_U tamed by ω. Then allowing arbitrarily small pertubations of J_U near ∂U we can extend J_U to a $C^{k+\alpha}$ almost complex structure J on N tamed by ω.*

Proof. Without loss of generality we can suppose ∂U is a compact codimension one submanifold of N. Let $T_{\partial U}$ be a tubular neighborhood of ∂U. Parametrize the fibers of $T_{\partial U} \cap U$ by the unit interval so that the map

$$\rho\colon [0,1] \times \partial U \to T_{\partial U} \cap U$$

is the identity on $\{1\} \times \partial U$. After choosing a metric on N there is a "canonical" C^∞ almost complex structure J_ω on N tamed by ω [16]. Consider the almost complex structures J_U and J_ω restricted to $T_{\partial U} \cap U$ and choose a path $\gamma\colon [0,1] \to \mathscr{J}_{T_{\partial U} \cap U}$ in the space of almost complex structures on $T_{\partial U} \cap U$ tamed by ω, such that $\gamma(0) = J_U$ and $\gamma(1) = J_\omega$. Then for $x \in N$, set

$$J(x) = \begin{cases} J_U(x), & x \in U - T_{\partial U}, \\ \gamma(t)(x), & x \in \rho(\{t\} \times \partial U), \\ J_\omega(x), & x \in N - U. \end{cases}$$

Moreover for suitable choice of γ, J can be made $C^{k+\alpha}$ smooth. q.e.d.

We now consider the special case of Example 7.7 with $r = 1$ (i.e., $N = S^2 \times V$).

Proposition 9.3 [6, 0.3.A]. *Let U be an open submanifold of $N = S^2 \times V$ and let J_U be a $C^{k+\alpha}$, $k \geq 1$, almost complex structure on U which is tamed by the symplectic form $\omega\,|_U = \omega_1 \oplus \omega_2\,|_U$. Then for every point $u \in U$ there is a connected J_U-holomorphic curve C which passes through u and for which $\int_C \omega\,|_U \leq \int_{S^2} \omega_1$.*

Proof. Extend J_U to a $C^{k+\alpha}$ almost complex structure J on $S^2 \times V$ tamed by $\omega_1 \oplus \omega_2$. By Theorem 9.1 there is a J-holomorphic map $f\colon S^2 \to S^2 \times V$ such that $f(s_1) = u$ and $\int_{f(S^2)} \omega = \int_{S^2} f^*(\omega) = \int_{S^2} \omega_1$. For C take the connected component of u in the intersection $f(S^2) \cap U$. q.e.d.

Let $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ be coordinates on $\mathbb{R}^{2n}$ and consider $\mathbb{R}^{2n}$ as a symplectic manifold with symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.

Theorem 9.4 (*The symplectic rigidity of the round ball* [6, *Corollary* 0.3.A]). *Let $B_R \subset \mathbb{R}^{2n}$ denote the open round ball of radius R. Let $U \subset \mathbb{R}^{2n}$ be an open subset of the ε-neighborhood of the subspace $\mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$ defined by $x_n = y_n = 0$. If $(B_R, \omega_0\,|_{B_R})$ and $(U, \omega_0\,|_U)$ are symplectically diffeomorphic, then $R \leq \varepsilon$.*

Proof. Suppose $(B_R, \omega_0\,|_{B_R})$ and $(U, \omega_0\,|_U)$ are symplectically diffeomorphic and denote the diffeomorphism by $h\colon B_R \to U$. Without loss of generality we can suppose that U has compact closure, since for any δ, $R > \delta > 0$, $h(B_{R-\delta}) = U_\delta \subset U$ has compact closure and is symplectically diffeomorphic to $B_{R-\delta}$. The theorem then implies $R - \delta \leq \varepsilon$ for all $\delta > 0$ and hence $R \leq \varepsilon$. It follows that U can be symplectically embedded into $(S^2 \times T^{2n-2}, \omega_1 \oplus \omega_2)$ for the 2-form ω_1 on S^2 such that $\int_{S^2} \omega_1 = \pi\varepsilon^2$ and for the 2-form $\omega_2 = N\pi\varepsilon^2 \sum_{i=1}^{n-1} dx_i \wedge dy_i$ on the torus $T^{2n-2} = \mathbb{R}^{2n-2}/\mathbb{Z}^{2n-2}$ where N is a sufficiently large integer. The symplectic form ω_2 satisfies the integrality condition (7.8) and so for any almost complex structure J on U tamed by $\omega_1 \oplus \omega_2$ and any point $u \in U$ there is a J-holomorphic curve C in U passing through u such that $\int_C \omega_1 \oplus \omega_2 \leq \pi\varepsilon^2$. As B_R is symplectically diffeomorphic to U, for any almost complex structure J on B_R tamed by ω_0 there is a J-holomorphic curve C in B_R passing through the center of B_R such that $\int_C \omega_0 \leq \pi\varepsilon^2$. But for the standard complex structure on $\mathbb{C}^n = \mathbb{R}^{2n}$, every holomorphic curve through the center of B_R has area $= \int_C \omega_0 \geq \pi R^2$. Hence $R \leq \varepsilon$.

Definition 9.5. The symplectic radius of a symplectic manifold N is the least upper bound of the radii of the balls $B_R \subset \mathbb{R}^{2n}$ which admit symplectic imbeddings into N.

Example 9.6. Let $D_{R_i} \subset \mathbb{R}^2$, $i = 1, 2$, be the open round disk of radius R_i in $\mathbb{R}^2$ with symplectic form $dx \wedge dy$. Then the symplectic radius of $N = D_{R_1} \times D_{R_2}$ is $\min(R_1, R_2)$.

Corollary 9.7. *The symplectic manifolds $D_{R_1} \times D_{R_2}$, $R_1 \leq R_2$, and $D_{R_3} \times D_{R_4}$, $R_3 \leq R_4$, are symplectically diffeomorphic if and only if $R_1 = R_3$ and $R_2 = R_4$.*

Proof. If $D_{R_1} \times D_{R_2}$ and $D_{R_3} \times D_{R_4}$ are symplectically diffeomorphic, then $R_1 = R_3$. But volume$(D_{R_i} \times D_{R_j}) = \pi^2 R_i^2 R_j^2$, $(i, j) = (1, 2)$ or $(3, 4)$, is a symplectic invariant and hence $R_2 = R_4$.

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